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THE PRINCIPIA'S THEORY OF THE MOTION OF THE LUNAR APSE

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Summaries

An analysis of Newton's theory of the lunar apsidal motion in the Principia *shows an inadequacy for which he attempted to compensate by adjusting his numerical assumptions.*

Анаииз Нютонской теории по лунному апсидному дъижению ь *Principia* покадыьает недостаточность, которию он пробоьаи компенсироьать по регулироьанию сьоих чисиоьых предпоиожеий.

In the *Principia* there is no satisfactory treatment of the lunar apsidal motion. It may also be shown that Newton was not entirely forthright about the difficulties he had with this problem. In this paper we shall examine the theory of the lunar apsidal motion which can be extracted from the *Principia*. This theory, rather than the unpublished lemmas and calculations found in the Portsmouth Collection [1], was what confronted and troubled Euler, Clairaut, and D'Alembert [2]. The source of the theory's inadequacy will here be indicated.

In the *Principia*, Proposition III, Book III, we find the following statement:

> *The action of the sun, attracting the moon from the earth, is nearly as the moon's distance from the earth; and therefore (by what we have shown in Cor. II, Prop. XLV, Book I) is to the centripetal force of the moon as 2 to 357.45, or nearly so; that is, as 1 to 178 29/40.* [Newton 1934, 408]

Why does Newton here refer to Cor. II, Prop. XLV? There, he supposes the "foreign force [i.e. the solar perturbing force acting along the earth-moon radius] to be 357.45 times less than the other force with which the body revolves in the ellipse." [Newton 1934, 147] In other words, the ratio of solar perturbing force to terrestrial centripetal force (acting on the moon) found in Cor. II has been doubled in Prop. III. And this is very convenient: the ratio of 1 to 357.45 produces only about half the observed motion of the lunar apse (an assertion Newton adds to Cor. II in the third edition [Newton 1972 Vol. 1, Koyré/Cohen note 30, 242]), whereas the doubled ratio of 1 to 178 29/40 produces a good approximation of the rate of apsidal advance.

It appears that at most one of these ratios can be correct. And it seems extremely odd that Newton would choose a ratio for use in Cor. II which would predict an erroneous value for the apogeal motion: he was at liberty to choose any ratio at all. We shall see that he did choose the correct ratio in the Corollary; so that the erroneous prediction indicated an inadequacy in his lunar theory. It seems, then, that Newton doubled this ratio in Prop. III in order to obtain a correct prediction.

Newton's derivation of the value of the ratio in question is to be found in Prop. XXV, Book III: "To find the forces with which the sun disturbs the motions of the moon." [Newton 1934, 440].

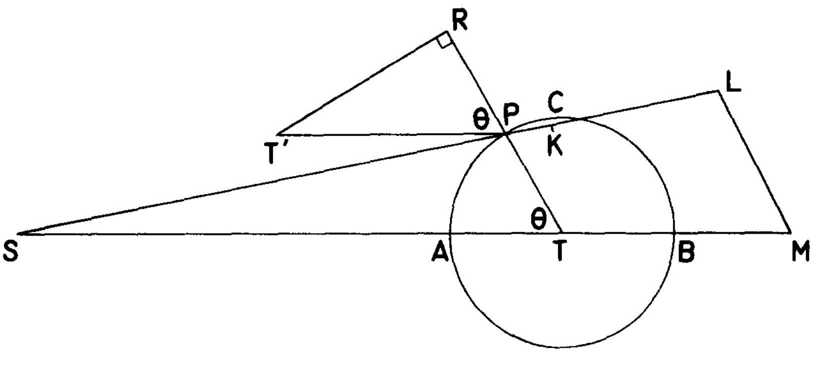

FIGURE I

Using Figure 1, familiar from Prop. LXVI, Book 1, Newton determines quantitative values for the perturbing force of the sun on the moon. For a given position P of the moon in its orbit ABC around the earth T, the components of the solar perturbing force are $\vec{LM}$ and $\vec{MT}$. He sets up the proportion

$$\vec{LM} : \text{centripetal force of } T \text{ at distance } \overline{PT} :: 1 : 178\ 29/40,$$

derived from Cor. XVII, Prop. LXVI, where it is shown that the mean value of $\vec{LM}$ (expressed by $\vec{PT}$) is to the centripetal force of the earth of distance $\overline{PT}$ as the square of the periodic time of the moon about the earth to the periodic time of the earth about the sun (or as 1 to 178 29/40). [*ibidem,* 184-185]

Then by Prop. IV, Book III, Newton shows that

$\vec{LM}$: centripetal force of T at surface of T :: 1 : 638092.6, "hence, by the proportion of the lines $\overline{TM}$, $\overline{ML}$, the force $\vec{MT}$ is also given; and these are the forces with which the sun disturbs the motion of the moon." [*ibidem*, 441] If Newton did not think his theory correct in detail, not simply in qualitative outline, it is difficult to understand why he would calculate a numerical value for $\vec{LM}$ (and implicitly for $\vec{MT}$).

But now we must squarely face the problem of the ratio of 1 to 178 29/40. For even if the mean value of $\vec{LM}$ is to the earth's centripetal force at distance $\overline{PT}$ as 1 is to 178 29/40, still the entire radial perturbing force is not accounted for. $\vec{MT}$ may be decomposed into $\vec{MT}\cos\theta$ and $\vec{MT}\sin\theta$, where θ is the angle STP. In Figure 1, if $\overline{PT'}$ is equal and parallel to $\vec{MT}$, then $\vec{MT}\cos\theta = \vec{PR}$, and $\vec{MT}\sin\theta = \vec{RT'}$. Thus, the total radial perturbing force will be the vector sum of $\vec{LM}$ and $\vec{MT}\cos\theta$, or of $\vec{PT}$ and $\vec{PR}$. (Note that the addends are oppositely directed.) We therefore see why Newton used the undefined term "action" in Prop. III: this action is $\vec{LM}$ (only part of the total radial perturbing force), and, as we shall see, it "is nearly as the moon's distance from the earth."

At this point it is very important to be clear about what has been shown and what remains to be shown. In Prop. XXV, Newton appeals to Cor. XVII, Prop. LXVI, in arguing that the mean value of $\vec{LM}$ is $\vec{PT}$. But there is no real demonstration in this corolllary:

> *Since the line $\overline{LM}$ becomes greater and sometimes less than the radius $\overline{PT}$, let the mean quantity of the force $\vec{LM}$ be expressed by that radius $\overline{PT}$.*
> [*ibidem*, 184]

Surely, a better proof is needed. [3]

Considering Figure 1, we have by similar triangles $\overline{LM} = (\overline{SM}/\overline{ST})(\overline{PT})$. Now as S is taken farther from T and P, the ratio $\overline{SM}/\overline{ST}$ approaches unity, so that $\overline{LM}$ becomes more nearly equal to $\overline{PT}$. Furthermore, $\overline{LM} > \overline{PT}$ if and only if $\overline{SM} > \overline{ST}$; but $\overline{SM} > \overline{ST}$ if and only if $\overline{SP} < \overline{ST}$. On the other hand, $\overline{LM} < \overline{PT}$ if and only if $\overline{SM} < \overline{ST}$; but $\overline{SM} < \overline{ST}$ if and only if $\overline{SP} > \overline{ST}$. Since $\overline{LM}$ approximates $\overline{PT}$ at the quadratures, $\overline{SP} > \overline{ST}$ for about half the orbit, and $\overline{SP} < \overline{ST}$ for the other half, Newton concludes that $\overline{LM}$ has the mean value $\overline{PT}$.

A more rigorous proof that $\overline{PT}$ is the mean value of $\overline{LM}$ is as follows. By construction, $(\overline{SK}^2/\overline{SP}^2) = (\overline{SL}/\overline{SK})$; therefore, $(\overline{SK}^3/\overline{SP}^3) = (\overline{SL}/\overline{SP})$. By similar triangles, $(\overline{SL}/\overline{SP}) = (\overline{LM}/\overline{PT})$; therefore, $(\overline{SK}^3/\overline{SP}^3) = (\overline{LM}/\overline{PT})$. Let $\overline{PT} = r$ and $\overline{SK} = \overline{ST} = R$. By the cosine law, $\overline{SP}^2 = R^2 + r^2 - 2Rr\cos\theta$. But we also have $\overline{LM} = (\overline{PT})(\overline{SK}^3/\overline{SP}^3) = (R^3 r/\overline{SP}^3) = R^3 r/(R^2 + r^2 - 2Rr\cos\theta)^{3/2}$.

Then $LM = r/(1 - \frac{2r}{R}\cos\theta)^{3/2}$, where r^2/R^2 ($\approx 1/300^2$) has been ignored. Or, $\overline{LM} \simeq r/(1 - \frac{3r}{R}\cos\theta)$, by the binomial theorem, where terms after the second have been ignored. Therefore, $LM \simeq r(1 + \frac{3r\cos\theta}{R})$, and the average of LM over one revolution is approximately r.

So far we have shown that the mean value for $\vec{LM}$ is measured by $\overline{PT}$ and *therefore*, that LM :centripetal force of T at distance PT :: 1 : 178 29/40. Thus we have found that $\vec{LM}$ is proportional to r and that it *adds* to the effect of the earth's centripetal force--a result which would cause the apse to regrede (contrary to observation).

As for $\vec{MT}\cos\theta$, or $\vec{PR}$, we recall that it acts to oppose $\vec{LM}$ and the earth's centripetal force. From Cor. VI, Prop. LXVI, we have

$$\overline{KL} = \frac{\overline{SK}(\overline{SK}^2 - \overline{SP}^2)}{\overline{SP}^2} = \frac{\overline{SK}(\overline{SK} + \overline{SP})\overline{PK}}{\overline{SP}^2}$$

Therefore, $(\overline{KL}/\overline{PK}) = (\overline{SK})(\overline{SK} + \overline{SP})/(\overline{SP}^2)$. Again let S recede from T and P; ultimately, $SK = SP$. Thus, in the limit, $(KL/PK) = 2$. Then we have $\overline{KL} = 2PK$, or $KL + PK = PL = 3PK$. In the same ultimate situation we find $\overline{PK} = \overline{PT}\cos\theta$, and $\overline{PL} = MT$. Therefore, $\overline{MT} = 3\overline{PK} = 3r\cos\theta$, and $\overline{PR} = 3r\cos^2\theta = (3r/2)(1 + \cos 2\theta)$. Then the average value of $\overline{PR}$ over one revolution is $3r/2$. Since the force $\vec{PR}$ is subtractive, the net effect of the centripetal force in moving the apse will be given by:

$$(\vec{LM})_{av} - (\vec{PR})_{av} = r - (3r/2) = - r/2.$$

So when Newton finds $\vec{LM}$ to be $1/178\frac{29}{40}$ of the earth's force, he halves it and reverses its sign to get the net force on the moon due to the sun. That is, the total radial perturbing force at P is to the centripetal force of the earth at P as 1 is to 357.45. Hence, the total centripetal force has the form: $F = (b/r^2) - (cr) = (br - cr^4)/r^3$. But this form is exactly the subject of Example III, Prop. XLV, allowing us to find the motion of the line of apsides.

It is clear, then, that the ratio of the forces given in Cor. II corresponds to the real case, and the doubled ratio given in Prop. III still requires explanation. Certainly Newton provides no proof that the ratio should be doubled. Furthermore, even if $\vec{LM}$ did measure the total radial perturbing force, it would add to the earth's centripetal force and cause regression of the apse. The true measure of the radial perturbing force must be oppositely directed.

D. T. Whiteside has suggested to me that Prop. III can be read as follows: Prop. XLV *would* yield the correct apsidal advance *if* the sun's radial perturbing force is doubled and oppositely directed and all other possibilities of increasing apsidal advance (better approximations of both components of the perturbing force, the eccentricity of the lunar orbit, and even external, planetary attractions) are neglected. Alternately, the correct advance would result if all the latter possibilities (whose *individual* actions Newton could *not* measure) are taken to act conglomerately as though *together* they produced a radial force oppositely directed to and double that of the first approximation of the radial component. There is MS evidence for this interpretation:

> *Now according to the third example by which we have illustrated Cor. II, Prop. XLV, Book I, ... by the* [radial perturbing] *force the apogee covers a space of 1d 31′ 14″, while* [the moon] *progresses from apse to the same apse in individual revolutions. This force becomes to the centripetal force at the moon's mean distance as 1 to 357.45. But from observed individual revolutions of the moon, it covers twice that space, approximately 3d 3′; therefore, the perturbing force on the moon is to the centripetal force as 2 to 357.45, approximately.* [Newton MS]

As a matter of fact, no proof for the doubled ratio could be provided because most of the error lay in neglecting the transverse component of the perturbing force, $\vec{MT}\sin\theta$ in our notation. I have been able to show [1975b, esp. 138-142] on the basis of Clairaut's analytic treatment of the problem that neglect of the transverse component results in a calculated rate of apsidal advance only 62% of the observed value. The improvement from about 50% in Newton's treatment to 62% in Clairaut's may be attributed to a better series of approximations. That Newton had some inkling of the source of his theory's inadequacy is evident from his work in the unpublished lemmas found in the Portsmouth Collection *Catalogue* of 1888, in which he takes account of the transverse component. Dr. Whiteside [Newton 1975] has recently published a perspicuous analysis of this later theory. His analysis shows that in this theory the transverse component is nil--a result which probably underlies Newton's decision to retain the disingenuous treatment of Prop. III in the second and third editions.

NOTES

1. Isaac Newton, "On the Motion of the Apogee in an Elliptic Orbit of very small eccentricity" [Newton 1888].
2. For discussions of the historical questions involved, see [Chandler 1975b] and [C. B. Waff 1976]. For a discussion

of some of the philosophical issues, see [Chandler 1975a].

3. I am indebted to Curtis Wilson for suggestions in making the argument that follows in the text.

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